Statistics of the one-dimensional Riemann walk

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Abstract

The Riemann walk is the lattice version of the Lévy flight. For the one-dimensional Riemann walk of Lévy exponent $0 < \alpha < 2$ we study the statistics of the *support*, *i.e.* set of visited sites, after t steps. We consider a wide class of support related observables $M(t)$, including the number $S(t)$ of visited sites and the number $I(t)$ of sequences of adjacent visited sites. For $t \to \infty$ we obtain the asymptotic power laws for the averages, variances, and correlations of these observables. Logarithmic correction factors appear for $\alpha = \frac{2}{3}$ and $\alpha = 1$. *Bulk* and *surface* observables have different power laws for $1 \leq \alpha < 2$. Fluctuations are shown to be universal for $\frac{2}{3} \leq \alpha < 2$. This means that in the limit $t \to \infty$ the deviations from average $\Delta M(t) \equiv M(t) - \overline{M(t)}$ are fully described either by a single M independent stochastic process (when $\frac{2}{3} < \alpha \leq 1$) or by two such processes, one for the bulk and one for the surface observables (when $1 < \alpha < 2$).

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1 Introduction

The Lévy flight is a random walk in continuous space whose step size distribution has a power law tail and is therefore sometimes called a "Lévy [1] distribution". The ubiquity of such distributions has been emphasized by many authors, and is a consequence of the power law tail being invariant under convolution. Many interesting instances of the occurrence of Lévy distributions are given by Tsallis [2] and Tsallis *et al.* [3]. These range from applications in physics (superdiffusion, chaotic fluid flow) and engineering (leaking taps) through studies of the physiology of heart activity, all the way to descriptions [4] of fluctuations of financial markets.

A one-dimensional lattice version of the Lévy flight may be constructed as follows. Let a random walk consist of independent steps, and let the probability $p(\ell)$ for a displacement of ℓ lattice units in a single step be given by $p(0) = 0$ and

$$p(\ell) = A\,|\ell|^{-1-\alpha} \qquad (\ell = \pm 1, \pm 2, \dots) \tag{1.1}$$

Here $\alpha > 0$ is the *Lévy exponent.* Normalization of p implies that $A^{-1} = 2\,\zeta(1+\alpha)$ where ζ is the Riemann zeta function. This random walk was first studied by Gillis and Weiss [5] in 1970. It is called the *Riemann walk* by Hughes (Ref. [6], p. 154) and we will conform to that terminology. More generally we call *of Riemann type* any one-dimensional lattice walk whose $p(\ell)$ is asymptotically proportional to $|\ell|^{-1-\alpha}$ when $|\ell| \to \infty$.

Riemann type walks were reviewed in detail by Hughes [6]. Of particular interest is the exponent regime $0 < \alpha \leq 2$, where these walks have a mean square displacement per step, $\langle \ell^2 \rangle$, which is infinite. There then exists, at least for certain global walk features, a correspondence between simple random walk on a d-dimensional lattice and one-dimensional Riemann type walks of exponent $\alpha = 2/d$. In some ways the fraction $\frac{2}{\alpha}$ acts as the walk's *effective dimensionality.* But whereas analytical results for noninteger dimension d cannot be checked by computer simulations, the full continuum of α values is accessible to Monte Carlo studies.

Much interest has centered around the following question. Let there be a t step Riemann walk. Then what are the statistical properties of its *support* $\mathcal{S}(t)$, *i.e.,* of the set of sites that the walk has visited? There appears immediately an important difference between the exponent regimes $0 < \alpha < 1$ and $1 < \alpha < 2$. In the former regime the Riemann walk is transient [6, 7] and it is easy to show (see Sec. 2.5) that $\mathcal{S}(\infty)$ is a set of fractal dimension $d_{\mathcal{S}} = \alpha$. In the latter case the Riemann walk is recurrent [6, 7], $\mathcal{S}(\infty)$ coincides with the full one-dimensional lattice, and $d_{\mathcal{S}} = 1$. The existing literature deals with the different question of finding the properties of $\mathcal{S}(t)$ for asymptotically t; the results reflect, nevertheless, the same distinction between $0 < \alpha < 1$ and $1 < \alpha < 2$. The borderline case $\alpha = 1$ is more subtle.

Gillis and Weiss [5] study the number $S(t)$ of distinct sites in the support. They find, among other results, that for $t \to \infty$ the average of this random variable behaves [5] as $\overline{S(t)} \sim t$ for $0 < \alpha < 1$ and as $\overline{S(t)} \sim t^{1/\alpha}$ for $1 < \alpha < 2$, where $\sim$ indicates asymptotic proportionality. For $\alpha = 1$ and $\alpha = 2$ power laws with logarithmic correction factors appear [5]. For $\alpha > 2$ the result $\overline{S(t)} \sim t^{1/2}$ is identical to that for the simple random walk in $d = 1$.

A recent extension of this work is due to Berkolaiko *et al.* [8]. Pursuing a question initially asked for the case of the simple random walk by Larralde *et al.* [9], these authors investigate the number $S_N(t)$ of distinct sites visited by N independent t step Riemann type walks all starting on the same lattice site. Again power laws appear, both for $t \to \infty$ at fixed N and for $N \to \infty$ at fixed t.

The present work extends the investigations of Gillis and Weiss into a different direction. We limit ourselves to the Riemann walk defined by Eq. (1.1), with α in the regime of greatest interest, that is, $0 < \alpha < 2$. Our results may be summarized under three headings.

1. Variance $\overline{\Delta S^2(t)}$. For any quantity $X(t)$ we will denote its instantaneous deviation from average by $\Delta X(t) \equiv X(t) - \overline{X(t)}$. Traditionally in this field the calculation of the average number $\overline{S(t)}$ of distinct sites visited has been followed by a calculation of the variance $\overline{\Delta S^2(t)}$ of that number. Thus, for the simple random walk $\overline{S(t)}$ was first calculated by Dvoretzky and Erdös [10] in 1951, and $\overline{\Delta S^2(t)}$ by Jain and Pruitt [11] in 1970. For the one-dimensional Riemann walk the present work supplements the 1970 results due to Gillis and Weiss [5] for $\overline{S(t)}$ by the corresponding ones for the variances $\overline{\Delta S^2(t)}$ in the regime $0 < \alpha < 2$.

2. Variables other than $S(t)$. We use the powerful generating function method (GFM), which was introduced into the field of random walks by Montroll [12] and Montroll and Weiss [13]. Overviews of this method are given by Weiss [7] and by Hughes [6]. The first calculation of a variance by the GFM, *viz.* that of $S(t)$ for the simple random walk, is due to Torney [14] in 1986.

In 1994 Coutinho *et al.* [15] performed Monte Carlo simulations of, among other things, the number of unvisited islands enclosed by the support of the t step simple random walk in two dimensions. This led Caser and Hilhorst [16] to analytically determine the asymptotic behavior of the average number of islands. Subsequently Van Wijland *et al.* [17, 18] developed a compact GFM based analytical scheme for calculating simultaneously the averages, variances, and correlations of a large class of observables characteristic of the support, generically denoted by the symbol $M(t)$. In $d = 2$ this class includes also the total boundary length of the support, and in $d = 3$ its surface area and Euler index.

Here we bring this scheme to bear on the one-dimensional Riemann walk. The support of this walk consists of alternating sequences of visited and unvisited sites. Among the most prominent members of the class of observables

$M(t)$ is, next to $S(t)$, the number of visited sequences, that we will denote by $I(t)$. Islands in $d=1$ just are unvisited sequences enclosed by the support, of which there are $I(t)$–1; the support furthermore has $2I(t)$ boundary sites (= visited sites adjacent to an unvisited one). Table I summarizes our results for the asymptotic laws of the averages, variances, and correlations involving $S(t)$ and $I(t)$. Beyond their intrinsic interest these laws may serve in heuristic arguments in reaction–diffusion processes, *e.g.*, to estimate the trapping probability of an atom that diffuses in a random absorbing environment, or the effective reaction rate between two diffusing species. We defer further comments to Sec. 7.

	$0<\alpha<\frac{2}{3}$	$\alpha=\frac{2}{3}$	$\frac{2}{3}<\alpha<1$	$\alpha=1$	$1<\alpha<2$
$\overline{S(t)}$ $\overline{I(t)}$	t	t	t	$t\,\log^{-1}t$ $t\,\log^{-2}t$	$t^{\frac{1}{\alpha}}$ $t^{\frac{2}{\alpha}-1}$
$\overline{\Delta S^2(t)}$ $\overline{\Delta S(t)\Delta I(t)}$ $\overline{\Delta I^2(t)}$	t	$t\log t$	$t^{4-\frac{2}{\alpha}}$	$t^2\log^{-4}t$ $t^2\log^{-5}t$ $t^2\log^{-6}t$	$t^{\frac{2}{\alpha}}$ $t^{\frac{3}{\alpha}-1}$ $t^{\frac{4}{\alpha}-2}$

Table I. Leading asymptotic behavior as $t\to\infty$ of the averages, variances, and correlation of $S(t)$ and $I(t)$ in different regimes of the Lévy exponent α. The exact prefactors of the asymptotic laws are given in the text. The results for $\overline{S(t)}$ are due to Gillis and Weiss [5]; the result for $\overline{\Delta S^2(t)}$ in the range $0<\alpha<\frac{2}{3}$ follows from the theorem of Jain, Orey, and Pruitt (see Hughes [6], p. 344); all others are new.

3. Universality of fluctuations. The deviations from average $\Delta S(t)$,$\Delta I(t)$, …,$\Delta M(t)$,… are randomly time-dependent variables that one would *a priori* expect to exhibit some degree of correlation. One calls these fluctuations *universal* – by lack of a better name – when in the limit $t\to\infty$ all $\Delta M(t)$ are asymptotically equal (up to a proportionality constant) to a *single* M independent stochastic process. For the simple random walk universality was shown to hold in dimensions $d=2$ [17] and $d=3$ [18], and not to hold in $d=4,5,\ldots$. For the $d=1$ Riemann walk we find that universality holds in the exponent regime $\frac{2}{3}\leq\alpha<2$, but not for $0<\alpha<\frac{2}{3}$. A novelty with respect to the case of the simple random walk is that for $1<\alpha<2$ not a single, but *two* M independent processes are needed to describe the universal fluctuations: one applies to bulk and the other to surface observables. The precise statements are given in Sec. 6.

This article is set up as follows. Sec. 2 describes those elements of our analysis that are common to the full exponent interval $0<\alpha<2$. Secs. 3 and 4 deal more in particular with the exponent regimes $0<\alpha<1$ and $1<\alpha<2$, respectively, and derive the asymptotic behavior of averages, variances, and correlations. In Sec. 5 we do the same for the exceptional values $\alpha=\frac{2}{3}$ and $\alpha=1$. In Sec. 6 we discuss the universality properties. In Sec. 7 we provide some additional interpretation of our results and conclude.

2 Observables, averages, and correlations

2.1 Observables $M(t)$

Our analysis is based on first writing quantities of interest in terms of the field $m(x,t)$ of "complementary occupation numbers" defined by $m(x,t) = 1$ if site x has not yet been visited at time t, and $m(x,t) = 0$ otherwise. The expressions of S and I in terms of m are

$$S(t) = \sum_{x=-\infty}^{\infty} [1 - m(x,t)] \tag{2.1}$$

$$I(t) = \sum_{x=-\infty}^{\infty} m(x,t)[1 - m(x+1,t)] \tag{2.2}$$

S and I are representatives of a general class of "observables" M that are sums on x of a summand to which each lattice site contributes a factor m $1-m$, or 1, *i.e.,* the summand tests for the presence of a specific pattern of visited (*"black"*) and unvisited (*"white"*) sites. The following slightly more abstract characterization of the M will be needed. Let $A = \{a\}$ be a finite set of distinct nonnegative integers a, such that either $A = \emptyset$ or, if not, A includes the element $a = 0$. The general observable $M(t)$ that we will consider is

$$M(t) = \sum_{x=-\infty}^{\infty} \sum_{A} \mu_A \prod_{a\in A} m(x+a,t) \tag{2.3}$$

where for $A = \emptyset$ the product is equal to unity and where $\{\mu_A\}$ is a set of numerical coefficients characteristic of M. When their M dependence needs to be indicated we will write $\mu_A[M]$. Eqs. (2.2) and (2.1) show that $S(t)$ and $I(t)$ are of the form of Eq. (2.3) with only two nonzero coefficients, as shown in Table II.

A	$\mu_A[S]$	$\mu_A[I]$	$\mu_A[S_1]$	$\mu_A[I_1]$
$\emptyset$	1			
$\{0\}$	-1	1		1
$\{0,1\}$		-1		-2
$\{0,2\}$			1	
$\{0,1,2\}$			-1	1

Table II. Coefficients $\mu_A[M]$ for the four observables $M = S, I, S_1, I_1$ defined in the text; entries not shown are zero. The coefficients in each column add up to zero.

Two further examples of observables of type (2.3) are the total number $S_1(t)$ of visited sequences consisting of only a single site, and the total number $I_1(t)$ of single-site unvisited sequences. Their coefficients μ_A involve sets

A of up to three elements; they are easily determined and have also been listed in Table II.

The following remarks, important for later, are verified without much effort. The coefficient $A_\emptyset$ is nonzero if and only if M is built up exclusively out of factors m. Since these correspond to visited sites, that make up the "bulk" of the support, we will call an M of this type a *bulk* observable. Observables built up exclusively out of factors m do not occur, since their expectation value on an infinite lattice is infinite. Hence the remaining observables refer to patterns consisting of both visited and unvisited sites, and we will therefore call them *surface* observables. [In the terminology of Refs. [17, 18] these are *"black"* and *"black-and-white"* observables. They might also be called *"S-like"* and *"I-like"*, respectively.] The distinction between these two subclasses will play a role only in the exponent regime $1 \leq \alpha < 2$.

2.2 Basic formulas for averages and correlations

In this work we will first evaluate the $t \to \infty$ behavior of the averages $\overline{M(t)}$. Then we turn to the covariance matrix $\overline{\Delta M(t) \Delta M'(t)}$, where $M'(t)$ is a second observable with coefficients μ'_A. Although the authors of Refs. [17] and [18] deal with the simple random walk, the larger part of their formal developments also holds for the Riemann walk.

The averages $\overline{M(t)}$ and $\overline{M(t)M'(t)}$ can be obtained as follows [17, 18]. Let $G(x,t)$ be the Green function of the one-dimensional Riemann walk, that is, the probability for a walker starting at the origin to occupy site x after t steps. Let $\hat{G}(x,z) = \sum_{t=0}^{\infty} z^t G(x,t)$ denote its generating function and let $\mathbf{G}_A(z)$ be the $|A| \times |A|$ matrix of elements $\hat{G}(a-a', z)$ with $a,\, a' \in A$. From this matrix one constructs the "inverse sum" $\mathsf{G}_A(z)$ defined by

$$\mathsf{G}_A^{-1}(z) = \sum_{a,a' \in A} [\mathbf{G}_A^{-1}(z)]_{aa'} \tag{2.4}$$

These scalars satisfy certain elementary relations stated in Appendix A as Properties 1–3. Two functions $C_M(z)$ and $C_{MM'}(z)$ are defined in terms of the $\mathsf{G}_A(z)$ according to

$$C_M(z) = \sum_{A \neq \emptyset} \mu_A \frac{1}{\mathsf{G}_A(z)} \tag{2.5}$$

$$C_{MM'}(z) = \sum_{A \neq \emptyset} \sum_{B \neq \emptyset} \mu_A \mu'_B \sum_{r=-\infty}^{\infty} \Big[\frac{1}{\mathsf{G}_{A \cup (r+B)}(z)} - \frac{1}{\mathsf{G}_A(z)} - \frac{1}{\mathsf{G}_B(z)} \Big] \tag{2.6}$$

Here $A \cup (r+B)$ denotes the union of the set B, translated by r, and A. The averages $\overline{M(t)}$ and $\overline{M(t)M'(t)}$ are then obtained as [17, 18]

$$\overline{M(t)} = -\frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z^{t+1}} \frac{1}{(1-z)^2} C_M(z) \tag{2.7}$$

$$\overline{M(t)M'(t)} = -\frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}z}{z^{t+1}} \frac{1}{(1-z)^2} C_{MM'}(z) \tag{2.8}$$

where the integrations are counterclockwise around the origin.

The identities (2.4)–(2.8) are fundamental to random walk theory; they hold for any translationally invariant random walk, whether on a finite lattice with periodic boundary conditions or on an infinite lattice. They allow for the calculation, in a very compact way, of many known and new results.

Special cases. When $M(t) = S(t)$, the following simplifications occur. The sums on A and on B in Eqs. (2.5) and (2.6) then have only the single term with $A = \{0\}$ and $B = \{0\}$, respectively. Furthermore $\mathsf{G}_{\{0\}}(z) = \hat{G}(0, z)$, the matrix $\mathbf{G}_{\{0\}\cup(r+\{0\})}(z)$ is two by two, and an easy calculation leads to $\mathsf{G}_{\{0\}\cup(r+\{0\})}(z) = \frac{1}{2}(\hat{G}(0, z) + \hat{G}(r, z))$. When $M(t) = I(t)$, the sum in Eq. (2.5) involves $\mathsf{G}_{\{0\}}(z)$ and $\mathsf{G}_{\{0,1\}}(z) = \frac{1}{2}(\hat{G}(0, z) + \hat{G}(1, z))$. The sums on A and B in Eq. (2.6) then lead to four terms, which may be evaluated with a little more effort.

2.3 Limit $t \to \infty$ and scaling limit

Explicit evaluation of the general expressions (2.4) -(2.8) is limited in practice by the calculation of the inverse sums G_A, which require the inversion of a matrix of dimension $|A|$. Similarly, evaluation of $\mathsf{G}_{A\cup(r+B)}$ is an inversion problem of dimension $|A| + |B|$ (when A and $r + B$ have an empty intersection). It turns out that the sum on r in Eq. (2.6) can be performed only in the scaling limit

$$z \to 1, \quad |r| \to \infty \qquad \text{with} \quad \xi = r(1 - z)^{\frac{1}{\alpha}} \quad \text{fixed} \tag{2.9}$$

Finally, it will be possible to evaluate the integrals in Eqs. (2.7) and (2.8) only asymptotically for $t \to \infty$, a limit already implied by Eq. (2.9).

In order to prepare for these limits we rewrite the preceding expressions as follows. Using the simplified notation $G_0(z) = \hat{G}(0, z)$ we split the generating function $\hat{G}(x, z)$ up according to

$$\hat{G}(x, z) = G_0(z) - g(x, z) \tag{2.10}$$

In full analogy to $\mathbf{G}_A(z)$ we define $\mathbf{g}_A(z)$ as the matrix of elements $g(a-a', z)$ with a, $a' \in A$, and g_A^{-1} as the sum of all elements of $\mathbf{g}_A^{-1}$. Let now $\mathbf{J}$ be the square matrix of elements $J_{aa'} = 1$. Then

$$\mathbf{G}_A(z) = G_0(z)\,\mathbf{J} \;-\; \mathbf{g}_A(z) \tag{2.11}$$

and, by Property 1 of Appendix A,

$$\mathsf{G}_A(z) = G_0(z) - g_A(z) \tag{2.12}$$

This splitup will be useful for studying the $z \to 1$ behavior of $\mathsf{G}_A(z)$. Although $G_0(z)$ may $(1 \le \alpha < 2)$ or may not $(0 < \alpha < 1)$ diverge as $z \to 1$, the functions $g(x, z)$ and $g_A(z)$ remain finite in that limit.

We now turn to the inverse sum $\mathsf{G}_{A\cup(r+B)}$ constructed from the matrix $\mathbf{G}_{A\cup(r+B)}$. The dimension of this matrix is typically $|A|+|B|$. Let $\mathbf{J}^{AB}$ be the $|A|\times|B|$ matrix with all $J^{AB}_{ab}=1$. We then have (for $A\cap B=\emptyset$)

$$\mathbf{G}_{A\cup(r+B)}(z) = \begin{pmatrix} \mathbf{G}_A(z) & \hat{G}(r,z)\mathbf{J}^{AB} \\ \hat{G}(r,z)\mathbf{J}^{BA} & \mathbf{G}_B(z) \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{V} \\ \mathbf{V}^T & 0 \end{pmatrix} \tag{2.13}$$

where $\mathbf{V}$ is the matrix of elements

$$V_{a,r+b} = \hat{G}(r+b-a,z) - \hat{G}(r,z) \qquad a\in A,\ b\in B \tag{2.14}$$

and $\mathbf{V}^T$ is its transpose. The first matrix on the RHS of (2.13) has the form (A.4) of Appendix A. Applying Property 3 to that matrix we conclude that

$$\frac{1}{\mathsf{G}_{A\cup(r+B)}(z)} = \frac{\mathsf{G}_A(z)+\mathsf{G}_B(z)-2\hat{G}(r,z)}{\mathsf{G}_A(z)\mathsf{G}_B(z)-\hat{G}^2(r,z)} \;+\; \mathcal{O}(\mathbf{V}^2) \tag{2.15}$$

where we anticipate, and will have to show later, that $\mathbf{V}$ is small, that the correction terms are of order $\mathcal{O}(\mathbf{V}^2)$, and that they are negligible for our purpose.

Further analysis depends on the exponent α. We consider the two main regimes $0<\alpha<1$ and $1<\alpha<2$ in Secs. 3 and 4, respectively. The exceptional values $\alpha=\frac{2}{3}$ and $\alpha=1$ are discussed in Sec. 5.

2.4 Riemann walk Green function

All quantities of interest have been expressed above in terms of the Riemann walk Green function $\hat{G}(x,z)$. An elementary calculation yields

$$\hat{G}(x,z) = \int_{-\pi}^{\pi} \frac{\mathrm{d}q}{2\pi}\,\frac{\mathrm{e}^{-\mathrm{i}qx}}{1-z\lambda(q)} \tag{2.16}$$

$$\lambda(q) = \frac{1}{\zeta(1+\alpha)}\sum_{\ell=1}^{\infty}\ell^{-1-\alpha}\cos\ell q \tag{2.17}$$

The $q\to 0$ behavior of $\lambda(q)$ is crucial for the large scale features of the Riemann walk. It is known [5, 6] that

$$\lambda(q) = 1 - C_\alpha\,|q|^\alpha + \mathcal{O}(q^2) \qquad (q\to 0) \tag{2.18}$$

where for completeness we state the explicit expression

$$C_\alpha^{-1} = 2\zeta(1+\alpha)\,\Gamma(1+\alpha)\,/\,[\pi\sin(\alpha\pi/2)] \tag{2.19}$$

One finds by standard methods (see *e.g.* [6]) that in the limit $z\to 1$ the Green function in the origin $G_0(z)$ has the asymptotic expansion

$$G_0(z) = G_0(1) - B_\alpha(1-z)^{\frac{1}{\alpha}-1} + \mathcal{O}(1-z) \quad (0<\alpha<1;\alpha\neq\tfrac{1}{2}) \tag{2.20}$$

$$G_0(z) = \tfrac{1}{3}\log[c(1-z)^{-1}] + \mathcal{O}(1-z) \qquad (\alpha=1) \tag{2.21}$$

$$G_0(z) = A_\alpha(1-z)^{-1+\frac{1}{\alpha}} + \mathcal{O}(1) \qquad (1<\alpha<2) \tag{2.22}$$

where B_α and A_α are the constants

$$B_\alpha = -C_\alpha^{1/\alpha}/[2\sin(\pi/\alpha)] \qquad (\tfrac{1}{2} < \alpha < 1) \tag{2.23}$$

$$A_\alpha = 1/[2\alpha\, C_\alpha^{1/\alpha}\sin(\pi/\alpha)] \qquad (1 < \alpha < 2) \tag{2.24}$$

and c is a constant such that there is no $\mathcal{O}(1)$ term in Eq. (2.21). In Eqs. (2.20)–(2.22) and elsewhere we use the following convention. The symbol $\mathcal{O}(X)$ indicates terms that are *of order* X in the applicable limit ($X \to 0$ or $X \to \infty$); this however is not to say that all preceding terms are larger. Thus, the nonanalytic term in Eq. (2.20) is larger than the $\mathcal{O}(1-z)$ terms only for $\frac{1}{2} < \alpha < 1$. For $0 < \alpha < \frac{1}{2}$ it is present only as a correction to the $\mathcal{O}(1-z)$ terms; the expression for its coefficient B_α in that regime is different from Eq. (2.23) but will not be needed. For the borderline case $\alpha = \frac{1}{2}$, excluded from Eq. (2.20), we have $G_0(z) = G_0(1) - \frac{2}{\pi}(1-z)\log(1-z)^{-1} + \mathcal{O}(1-z)$; but this special nonanalytic behavior will stay subdominant everywhere in the remainder.

From Eqs. (2.16) and (2.18) one deduces that in the scaling limit (2.9)

$$\hat{G}(r,z) \simeq (1-z)^{\frac{1}{\alpha}-1} F(\xi) \qquad (0 < \alpha < 2) \tag{2.25}$$

where $\xi = r(1-z)^{\frac{1}{\alpha}}$ and $F(\xi)$ is the scaling function

$$F(\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \frac{\mathrm{e}^{-\mathrm{i}k\xi}}{1 + C_\alpha |k|^\alpha} \tag{2.26}$$

For $\xi \to 0$ it behaves as

$$F(\xi) \simeq 2\alpha\,\zeta(1+\alpha)/[\pi\sin(\alpha\pi)]\; \xi^{-1+\alpha} \quad (0 < \alpha < 1) \tag{2.27}$$

$$F(\xi) \simeq \tfrac{1}{3}\log\xi^{-1} \qquad (\alpha = 1) \tag{2.28}$$

$$F(\xi) = A_\alpha + \mathcal{O}(\xi^{\alpha-1}) \qquad (1 \le \alpha < 2) \tag{2.29}$$

In Secs. 4 and 5 we will also use the function $f(r,z)$ defined by

$$\hat{G}(r,z) = G_0(z) f(r,z) \qquad (1 \le \alpha < 2) \tag{2.30}$$

In the scaling limit one has $f(r,z) \simeq f(\xi) = F(\xi)/F(0)$ when $1 < \alpha < 2$.

2.5 Support at $t = \infty$

Whereas the remainder of this paper deals with the large t behavior, we briefly comment here on the structure of the support $\mathcal{S}(t)$ *at* $t = \infty$.

As is well-known [7, 6]), random walks are recurrent (are transient) if $G_0(1) = \infty$ (if $G_0(1) < \infty$). The Riemann walk of this work is recurrent for $1 \le \alpha < 2$, which means that all sites are visited with probability 1, and that *at* $t = \infty$ the support $\mathcal{S}(\infty)$ coincides with the full one-dimensional lattice.

For $0 < \alpha < 1$, however, the Riemann walk is transient, so that at $t = \infty$ the support $\mathcal{S}(\infty)$ will still be only a subset of the full lattice. We may estimate the average number of visited sites Σ_L between $x = -L$ and $x = L$ in $\mathcal{S}(\infty)$. According to standard random walk theory [7, 6]

$$\Sigma_L = \sum_{x=-L}^{L} \frac{\hat{G}(x,1)}{G_0(1)} \tag{2.31}$$

Upon substituting (2.16) in (2.31) one easily evaluates Σ_L for asymptotically large L, with the result that $\Sigma_L \sim L^\alpha$. It follows that the support $\mathcal{S}(\infty)$ has fractal dimension $d_{\mathcal{S}} = \alpha$.

3 Riemann walk of exponent $0 < \alpha < 1$

3.1 Averages

The large time behavior of $\overline{M(t)}$ comes from the behavior of $C_M(z)$, defined in Eq. (2.5), in the limit $z \to 1$. From Eq. (2.10) and the explicit expressions (2.16) and (2.17) it may be shown that $g(x,z) = g(x,1) + \mathcal{O}(1-z)$ for all $0 < \alpha < 1$. Hence

$$g_A(z) = g_A(1) + \mathcal{O}(1-z) \tag{3.1}$$

after which it follows from Eqs. (2.12), (2.22), and (3.1) that

$$\mathsf{G}_A(z) = \mathsf{G}_A(1) - B_\alpha(1-z)^{\frac{1}{\alpha}-1} + \mathcal{O}(1-z) \tag{3.2}$$

Inverting this relation and substituting in Eq. (2.5) gives

$$C_M(z) = -m_1 - B_\alpha m_2(1-z)^{\frac{1}{\alpha}-1} - B_\alpha^2 m_3(1-z)^{\frac{2}{\alpha}-2} + \ldots + \mathcal{O}(1-z) \tag{3.3}$$

where the m_n are determined by the coefficients μ_A of the observable M according to

$$m_n[M] = \sum_{A \neq \emptyset} \frac{\mu_A}{\mathsf{G}_A^n(1)} \qquad (n = 1, 2, \ldots\ ;\ \ 0 < \alpha < 1) \tag{3.4}$$

In Eq. (3.3) the dots stand for a power series in $(1-z)^{\frac{1}{\alpha}-1}$ and the number of nonanalytic terms between the zeroth and the first power of $1-z$ is equal to $n_\alpha \equiv \lceil \frac{\alpha}{1-\alpha} \rceil - 1$. That is, n_α is zero for $0 < \alpha < \frac{1}{2}$ and, as α goes up, jumps to $1, 2, 3, \ldots$ at $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$, respectively. After Laplace inversion we get for the average $\overline{M(t)}$ in the limit $t \to \infty$ the explicit result

$$\overline{M(t)} = m_1 t + \frac{B_\alpha}{\Gamma(3-\frac{1}{\alpha})} m_2 t^{2-\frac{1}{\alpha}} + \frac{B_\alpha^2}{\Gamma(4-\frac{2}{\alpha})} m_3 t^{3-\frac{2}{\alpha}} + \ldots + \mathcal{O}(1) \tag{3.5}$$

where the number of nonanalytic terms between the leading and the $\mathcal{O}(1)$ term is again equal to n_α.

3.2 Correlations

In this subsection we consider two – possibly equal – observables M and M', represented by sets of coefficients $\{\mu_A\}$ and $\{\mu'_A\}$, respectively, and wish to study their correlation. The starting point is Eq. (2.6) for $C_{MM'}(z)$, in which we substitute Eq. (2.15). Whereas $\mathsf{G}_A(z)$ and $\mathsf{G}_B(z)$ tend to finite values in the limit $z \to 1$, the Green function $\hat{G}(r,z)$ vanishes in that limit when taken with ξ fixed. This suggests that we expand in powers of $\hat{G}(r,z)$,

$$\frac{1}{\mathsf{G}_{A\cup(r+B)}(z)} - \frac{1}{\mathsf{G}_A(z)} - \frac{1}{\mathsf{G}_B(z)} = \sum_{n=1}^{\infty} C_{AB}^{(n)}(z)\hat{G}^n(r,z) \;+\; \mathcal{O}(\mathbf{V}^2) \tag{3.6}$$

with coefficients

$$C_{AB}^{(n)}(z) = \begin{cases} [\mathsf{G}_A(z)+\mathsf{G}_B(z)][\mathsf{G}_A(z)\mathsf{G}_B(z)]^{-\frac{n}{2}-1} & (n \text{ even}) \\ -2\,[\mathsf{G}_A(z)\mathsf{G}_B(z)]^{-\frac{n+1}{2}} & (n \text{ odd}) \end{cases} \tag{3.7}$$

The sum over space that occurs in Eq. (2.6) leads us to now consider the sums $\sum_r \hat{G}^n(r,z)$. From conservation of probability one finds that for $n=1$

$$\sum_r \hat{G}(r,z) = (1-z)^{-1} \tag{3.8}$$

For general n the calculation of $\sum_r \hat{G}^n(r,z)$ is slightly more laborious; after substituting Eq. (2.16) for $\hat{G}$ one finds by explicit expansion in powers of $1-z$ that for $z \to 1$ the sum on r behaves as

$$\sum_r \hat{G}^n(r,z) \simeq F_{\alpha,n}(1-z)^{-1+(n-1)(\frac{1}{\alpha}-1)} + \mathcal{O}(1) \qquad (0<\alpha<1; \alpha \neq 1-\tfrac{1}{n}) \tag{3.9}$$

For $n = 1 - \frac{1}{1-\alpha}$ (with $n = 1, 2, \ldots$) the sum on r instead behaves as $\log(1-z)$; this happens, in particular, for $n=3$ when $\alpha = \frac{2}{3}$, a case studied separately in Sec. 5.

We will now continue to consider the generic case. The nonanalytic term on the RHS of Eq. (3.9) dominates the $\mathcal{O}(1)$ term only for $n < \frac{1}{1-\alpha}$. In that case (3.9) follows just from the scaling form (2.22) of $\hat{G}$ and from the $\xi \to 0$ behavior (2.27) of $F(\xi)$. One then finds for the prefactor $F_{\alpha,n}$ the expression

$$F_{\alpha,n} = 2\int_0^{\infty} \mathrm{d}\xi\, F^n(\xi) \qquad (1-\tfrac{1}{n} < \alpha < 1) \tag{3.10}$$

For $0 < \alpha < 1 - \frac{1}{n}$ the expression for $F_{\alpha,n}$ is different and will not be needed. Eq. (3.10) shows that the main contribution to the sum on r comes from $\xi \sim 1$, that is, from $r \sim (1-z)^{-\frac{1}{\alpha}}$. For $n=1$ and $n=2$ the integral (3.10) yields the explicit results $F_{\alpha,1} = 1$, in agreement with Eq. (3.8), and $F_{\alpha,2} = (\frac{1}{\alpha} - 1)B_\alpha$, respectively. For $n > \frac{1}{1-\alpha}$ the sum on r draws its main contribution from the short distance (nonscaling) regime $r \sim 1$, and is of $\mathcal{O}(1)$ for $z \to 1$.

By successively substituting Eq. (3.7) in Eq. (3.6), neglecting the $\mathcal{O}(\mathbf{V}^2)$ terms in that equation – which is justified in Appendix B – , then substituting Eq. (3.6) in Eq. (2.6), expanding $\mathsf{G}_A(z)$ and $\mathsf{G}_B(z)$ according to Eq. (3.2), and using the $z \to 1$ behavior of $\sum_r \hat{G}^n(r,z)$ obtained in Eqs. (3.8) and (3.9) we find

$$\begin{aligned}
C_{MM'}(z) &= -2(1-z)^{-1} m_1 m_1' \\
&\quad - (1-z)^{\frac{1}{\alpha}-2} B_\alpha \Big(3 - \tfrac{1}{\alpha}\Big)(m_1 m_2' + m_2 m_1') \\
&\quad - (1-z)^{\frac{2}{\alpha}-3} \Big[\Big(4 - \tfrac{2}{\alpha}\Big) B_\alpha^2 (m_1 m_3' + m_3 m_1' + m_2 m_2') + 2F_{\alpha,3} m_2 m_2'\Big] \\
&\quad - \ldots + \mathcal{O}(1)
\end{aligned} \tag{3.11}$$

Here the dots stand for terms of order $(1-z)^{-1+k(\frac{1}{\alpha}-1)}$, with $k = 3, 4, \ldots$; and the m_n' are related to M' in the same way as the m_n are to M. For $t \to \infty$ we therefore find by substituting (3.11) in Eq. (2.8) and Laplace inverting

$$\begin{aligned}
\overline{M(t)M'(t)} &= m_1 m_1' t^2 \\
&\quad + \frac{B_\alpha}{\Gamma(3-\frac{1}{\alpha})}(m_1 m_2' + m_2 m_1') t^{3-\frac{1}{\alpha}} \\
&\quad + \Big[\frac{B_\alpha^2}{\Gamma(4-\frac{2}{\alpha})}(m_1 m_3' + m_3 m_1' + m_2 m_2') + \frac{2F_{\alpha,3}}{\Gamma(5-\frac{2}{\alpha})} m_2 m_2'\Big] t^{4-\frac{2}{\alpha}} \\
&\quad + \ldots + \mathcal{O}(t)
\end{aligned} \tag{3.12}$$

The successive terms in the above series all have one power of t more than the corresponding terms in the series (3.5) for $\overline{M(t)}$, and the number of nonanalytic terms between the leading and the $\mathcal{O}(t)$ term is once more equal to $n_\alpha - 1$. The product $\overline{M(t)}\ \overline{M'(t)}$, which follows from Eq. (3.5), now has to be subtracted from the series (3.12). This exactly cancels the terms in (3.12) proportional to t^2 and to $t^{3-\frac{1}{\alpha}}$ but leaves those proportional to $t^{4-\frac{2}{\alpha}}$ and of $\mathcal{O}(t)$. The $t^{4-\frac{2}{\alpha}}$ terms are leading only if $\frac{2}{3} < \alpha < 1$. Hence we find for the correlation between observables M and M' in the limit $t \to \infty$

$$\overline{\Delta M(t)\Delta M'(t)} \simeq \mathcal{B}_\alpha^2\, m_2 m_2'\, t^{4-\frac{2}{\alpha}} \tag{3.13}$$

valid for $\frac{2}{3} < \alpha < 1$, and in which

$$\mathcal{B}_\alpha^2 = \frac{2F_{\alpha,3}}{\Gamma(5-\frac{2}{\alpha})} + \frac{B_\alpha^2}{\Gamma(4-\frac{2}{\alpha})} - \frac{B_\alpha^2}{\Gamma^2(3-\frac{1}{\alpha})} \tag{3.14}$$

We have supposed here that m_2, $m_2' \neq 0$. The preceding analysis changes when either of these two coefficients vanishes. We do not know of any physically interesting examples where this happens, and do not pursue our analysis in this direction.

The borderline case $\alpha = \frac{2}{3}$ is considered in Sec. 5. In the interval $0 < \alpha < \frac{2}{3}$ the calculation of the present section applies, but with the result that

$$\overline{\Delta M(t)\Delta M'(t)} \simeq \kappa_{MM'}\, t \qquad (0 < \alpha < \tfrac{2}{3}) \tag{3.15}$$

in which the coefficient $\kappa_{MM'}$ has contributions from the $\mathcal{O}(1)$ terms in Eq. (3.5) and the $\mathcal{O}(t)$ terms in Eq. (3.12), and *does not factor* into an M and an M' dependent constant. This difference between Eqs. (3.13) and (3.15) is crucial for the phenomenon of universality discussed in Sec. 6.

4 Riemann walk of exponent $1 < \alpha < 2$

4.1 Averages

The calculation of $\overline{M(t)}$ starts again from the series (2.5) for $C_M(z)$. The calculation in the exponent regime $1 < \alpha < 2$ is different from that of the preceding section because now $G_0(z)$ diverges as $z \to 1$. Since $g_A(z)$ remains finite for $z \to 1$, this suggests that we use Eq. (2.12) and expand $\mathsf{G}_A(z)$ in powers of $g_A(z)/G_0(z)$. This yields

$$C_M(z) = \sum_{A \neq \emptyset} \mu_A \frac{1}{G_0(z)} \Big[\, 1 + \frac{g_A(z)}{G_0(z)} + \frac{g_A^2(z)}{G_0^2(z)} + \mathcal{O}(\frac{g_A^3}{G_0^3})\, \Big] \tag{4.1}$$

We now substitute in Eq. (4.1) the expansion (2.22) for $G_0(z)$ and use that $g_A(1)$ is finite. The result is a power series in $1 - z$ in which there appear coefficients that we denote again by m_n but that are defined for $1 \leq \alpha < 2$ as

$$m_n[M] = -\sum_{A \neq \emptyset} \mu_A\, g_A^n(1) \qquad (n = 0, 1, 2, \dots\,;\;\; 1 \leq \alpha < 2) \tag{4.2}$$

It will turn out that we need only the leading term, which is

$$C_M(z) \simeq \begin{cases} A_\alpha^{-1} m_0 (1-z)^{1-\frac{1}{\alpha}} & (m_0 \neq 0) \\[2ex] A_\alpha^{-2} m_1 (1-z)^{2-\frac{2}{\alpha}} & (m_0 = 0, m_1 \neq 0) \end{cases} \tag{4.3}$$

We pause to note that in the terminology of Sec. 2.1 the condition $m_0 \neq 0$ characterizes the bulk or *"S-like"* observables, and the condition $m_0 = 0$ the surface or *"I-like"* observables. This is the first equation where a difference appears between these two subclasses; in its analog, Eq. (3.5) of the preceding section, no such distinction appears.

Upon using Eq. (4.3) in Eq. (2.7) we obtain after an inverse Laplace transformation the asymptotic expansion of $\overline{M(t)}$ as $t \to \infty$,

$$\overline{M(t)} \simeq \begin{cases} [A_\alpha \Gamma(1 + \frac{1}{\alpha})]^{-1} m_0\, t^{\frac{1}{\alpha}} & (m_0 \neq 0) \\[2ex] [A_\alpha \Gamma(\frac{2}{\alpha})]^{-1} m_1\, t^{\frac{2}{\alpha}-1} & (m_0 = 0, m_1 \neq 0) \end{cases} \tag{4.4}$$

where the dots indicate terms of lower order in t.

4.2 Correlations

For the calculation of the correlation $\overline{M(t)M'(t)}$ via Eqs. (2.8) and (2.6) we have to return again to expression (2.15) for $1/\mathsf{G}_{A\cup(r+B)}(z)$, which is needed in Eq. (2.6). We use Eqs. (2.15), (2.12), and (2.30) to rewrite this quantity as

$$\frac{1}{\mathsf{G}_{A\cup(r+B)}(z)} = \frac{1}{G_0(z)} \frac{2(1-f(r,z)) - \frac{g_A(z)}{G_0(z)} - \frac{g_B(z)}{G_0(z)}}{1 - f^2(r,z) - \frac{g_A(z)}{G_0(z)} - \frac{g_B(z)}{G_0(z)} + \frac{g_A(z)g_B(z)}{G_0^2(z)}} + \mathcal{O}(\mathbf{V}^2) \tag{4.5}$$

The function $f(r,z)$ was defined in Eq. (2.30). In the scaling limit $f(r,z)$, $g_A(z)$, and $g_B(z)$ have finite limits, whereas $G_0(z)$ diverges. An expansion in inverse powers of $G_0(z)$ corresponds therefore to an expansion in ascending powers of $1-z$. Writing for short f, g_A, g_B, and G_0 when $f(r,z)$, $g_A(z)$, $g_B(z)$, and $G_0(z)$ are meant, we find after a straightforward calculation

$$\begin{aligned} C_{MM'}(z) \quad = \quad & \sum_{r=-\infty}^{\infty} \frac{1}{G_0}\frac{f}{1+f} \sum_{A,B\neq\emptyset} \mu_A\mu'_B \Big[\ 2 + (2+f)\,\frac{g_A+g_B}{G_0} \\ & + \frac{2-2f-f^3}{(1-f)(1+f)^2}\ \frac{g_A^2+g_B^2}{G_0^2} \\ & + \frac{2}{(1-f)(1+f)^2}\ \frac{g_Ag_B}{G_0^2}\ + \mathcal{O}(\frac{g_A^3}{G_0^3},\frac{g_B^3}{G_0^3})\ \Big] \end{aligned} \tag{4.6}$$

Let us write $m'_n = m_n[M']$ for the coefficients that characterize the observable M'. The two distinct cases described by Eq. (4.4) now lead to the following possibilities.

Case (i): $m_0 \neq 0$ and $m'_0 \neq 0$. In this case the leading term in the expression in brackets in Eq. (4.6) survives under the sum on A and B.

Case (ii): $m_0 = 0$, $m_1 \neq 0$, and $m'_0 \neq 0$. In this case in order to survive a term in the bracketed expression should contain at least one factor $g_A(z)$.

Case (iii): $m_0 = m'_0 = 0$ but $m_1 \neq 0$ and $m'_1 \neq 0$. In this case a term in order to survive must contain at least one factor $g_A(z)$ and one factor $g_B(z)$. Upon using in each of these cases for $G_0(z)$the expansion (2.22), passing to the scaling limit, and writing f for $f(\xi)$, we find that the result is

$$C_{MM'}(z) \simeq \begin{cases} -A_\alpha^{-1}(1-z)^{1-\frac{2}{\alpha}} f_{00}\, m_0m'_0 \\ -A_\alpha^{-1}(1-z)^{2-\frac{3}{\alpha}} f_{10}\, m_1m'_0 \\ -A_\alpha^{-1}(1-z)^{3-\frac{4}{\alpha}} f_{11}\, m_1m'_1 \end{cases} \tag{4.7}$$

in the three cases (i), (ii), and (iii), respectively; here the coefficients $f_{k\ell}$ represent the integrals

$$
\begin{aligned}
f_{00} &= 4\int_0^\infty \mathrm{d}\xi\ f(1+f)^{-1} \\
f_{10} &= 2\int_0^\infty \mathrm{d}\xi\ f(2+f)(1+f)^{-1} \\
f_{11} &= 4\int_0^\infty \mathrm{d}\xi\ f(1-f)^{-1}(1+f)^{-3}
\end{aligned} \tag{4.8}
$$

After substituting Eqs. (4.7) in Eq. (2.8) and carrying out the inverse Laplace transformation we find, in the limit $t\to\infty$,

$$
\overline{M(t)M'(t)} \simeq \begin{cases} \Gamma^{-1}(1+\frac{2}{\alpha})\ \ A_\alpha^{-1} f_{00}\, m_0 m_0'\ t^{\frac{2}{\alpha}} \\ \Gamma^{-1}(\frac{3}{\alpha})\ \ A_\alpha^{-2} f_{10}\, m_1 m_0'\ t^{\frac{3}{\alpha}-1} \\ \Gamma^{-1}(-1+\frac{4}{\alpha})\ \ A_\alpha^{-3} f_{11}\, m_1 m_1'\ t^{\frac{4}{\alpha}-2} \end{cases} \tag{4.9}
$$

respectively, for the three cases distinguished above. Upon combining these results with those of Section 4.1 one obtains, for $t\to\infty$,

$$
\overline{\Delta M(t)\Delta M'(t)} \simeq \begin{cases} \mathcal{B}_{\alpha 00}^2\, m_0 m_0'\, t^{\frac{2}{\alpha}} \\ \mathcal{B}_{\alpha 10}^2\, m_1 m_0'\, t^{\frac{3}{\alpha}-1} \\ \mathcal{B}_{\alpha 11}^2\, m_1 m_1'\, t^{\frac{4}{\alpha}-2} \end{cases} \tag{4.10}
$$

in which the coefficients $\mathcal{B}_{\alpha k\ell}$ are given by

$$
\begin{aligned}
\mathcal{B}_{\alpha 00}^2 &= A_\alpha^{-1}[f_{00}\Gamma^{-1}(1+\tfrac{2}{\alpha}) - A_\alpha^{-1}\Gamma^{-2}(1+\tfrac{1}{\alpha})] \\
\mathcal{B}_{\alpha 10}^2 &= A_\alpha^{-2}[f_{10}\Gamma^{-1}(\tfrac{3}{\alpha}) - A_\alpha^{-1}\Gamma^{-1}(1+\tfrac{1}{\alpha})\Gamma^{-1}(\tfrac{2}{\alpha})] \\
\mathcal{B}_{\alpha 11}^2 &= A_\alpha^{-3}[f_{11}\Gamma^{-1}(-1+\tfrac{4}{\alpha}) - A_\alpha^{-1}\Gamma^{-2}(\tfrac{2}{\alpha})]
\end{aligned} \tag{4.11}
$$

in the three cases (i), (ii), and (iii) defined above, respectively. We recall that the $f_{k\ell}$ on the RHS of Eq. (4.11) are given by Eq. (4.8) as integrals on $f(\xi)$, with $f(\xi)$ in turn given by Eq. (2.26).

5 Riemann walk of exponents $\alpha=\frac{2}{3}$ and $\alpha=1$

5.1 Exponent $\alpha=\frac{2}{3}$

In this special case $\overline{M(t)}$ is still given by Eq. (3.5). However, the calculation of $\overline{M(t)M'(t)}$ has to be reconsidered, as signalled by the fact that $F_{\alpha,3}$ in Eq. (3.11) diverges for $\alpha\to\frac{2}{3}^+$. In order to calculate $\sum_r \hat{G}(r,z)$ we cannot

now use Eq. (3.9). Instead we replace $\hat{G}(r,z)$ by its scaling form (2.22) but take into account that $|\xi|$ has a lower cutoff $|\xi| \sim \mathrm{cst}\times(1-z)^{\frac{2}{3}}$. This gives

$$\begin{aligned} \sum_r \hat{G}^3(r,z) &\simeq 2\int_{\mathrm{cst}\times(1-z)^{2/3}}^{\infty} \mathrm{d}\xi\, F^3(\xi) \\ &\simeq \tfrac{1}{2}\mathcal{C}^2 \log(1-z)^{-1} + \mathcal{O}(1) \qquad (z\to 1) \end{aligned} \tag{5.1}$$

where in the second step we used Eq. (2.24) and found for the coefficient the value $\mathcal{C}^2 = 2^{12}3^{-11/2}\pi^{-3}\zeta^3(\frac{3}{2})$. In this case, due to the $\log(1-z)$ in the equation above, $\overline{M(t)M'(t)}$ is larger than the product $\overline{M(t)}\ \overline{M'(t)}$ by a factor $\log t$, and determines by itself alone the final result, which reads

$$\overline{\Delta M(t)\Delta M'(t)} \simeq \mathcal{C}^2\, m_2 m_2'\, t\log t \qquad (t\to\infty) \tag{5.2}$$

This $t\log t$ behavior is the same as in the well-known case of the simple random walk in spatial dimension $d=3$ [18].

5.2 Exponent $\alpha = 1$

The case of Lévy exponent $\alpha = 1$ is subtler than the others. Since it is closely analogous to the simple random walk in dimension $d = 2$ [17], we will not present all steps in detail. Eq. (2.21) shows that for $\alpha = 1$ the Green function in the origin, $G_0(z)$, diverges as $z \to 1$. We can therefore expand $C_M(z)$ as a series in the same way as in Eq. (4.1). Since here again $g_A(z) = g_A(1) + \mathcal{O}(1-z)$, and in view of the logarithmic behavior (2.21), this series now leads to an expansion of $C_M(z)$ in inverse powers of $G_0(z)$. If the first nonzero term is of order $k+1$, then we have explicitly

$$C_M(z) = -m_k G_0^{-k-1}(z) - m_{k+1}G_0^{-k-2}(z) - m_{k+2}G_0^{-k-3}(z) + \dots \tag{5.3}$$

with the m_n defined by Eq. (4.2). The cases of physical interest have $k=0$ (bulk observables) or $k=1$ (surface observables), but it will be notationally convenient to keep k as a parameter. We will also refer to it as the *order* of M.

To find $C_{MM'}(z)$ we may still start from Eq. (4.5), but now the expansion of this equation runs differently. The reason is that for $z\to 1$ at fixed ξ the function $f(r,z)$ (defined by (2.30)) behaves as $F(\xi)/G_0(z)$ and so is of the same order as $g_A(z)/G_0(z)$ and $g_B(z)/G_0(z)$. We therefore have to perform a double expansion of the RHS of Eq. (4.5) in terms of on the one hand F/G_0 and on the other hand g_A/G_0 and g_B/G_0. The sum on r, which in the scaling limit becomes an integral on ξ, then leads to the appearance of coefficients $F_{1,n}$ defined as in Eq. (3.10) but with $\alpha = 1$. Special cases are $F_{1,1} = 1$ and $F_{1,2} = \frac{1}{3}$. Let m_k and $m'_{k'}$ be the first nonzero coefficients in the expansions of $C_M(z)$ and $C_{M'}(z)$, respectively. Then we find for $C_{MM'}(z)$,

retaining only the three leading order terms in the limit $z \to 1$,

$$
\begin{aligned}
C_{MM'}(z) \simeq -\frac{1}{(1-z)G_0^{k+k'+2}(z)}\Big[&2m_k m'_{k'} \\
&-G_0^{-1}(z)\Big(\tfrac{1}{3}(k+k'+2)a_2 m_k m'_{k'} - 2(m_k m'_{k'+1} + m_{k+1}m'_{k'})\Big) \\
&+G_0^{-2}(z)\Big(2F_{1,3}(k+1)(k'+1)a_3 m_k m'_{k'} \\
&\qquad - \tfrac{1}{3}(k+k'+3)a_2(m_k m'_{k'+1} + m_{k+1}m'_{k'}) \\
&\qquad + 2(m_k m'_{k'+2} + m_{k+1}m'_{k'+1} + m_{k+2}m'_{k'})\Big)\Big]
\end{aligned}
\tag{5.4}
$$

The inverse Laplace transforms of $C_M(z)$ and $C_{MM'}(z)$ may be found with the help of the explicit expression (2.21) for $G_0(z)$ and the integrals of Ref. [17]. We state only the explicit result for $\overline{M(t)}$, which is, for $t \to \infty$,

$$
\begin{aligned}
\overline{M(t)} \simeq \frac{3^{k+1}t}{\log^{k+1} ct}\Big[m_k \;+\; &\frac{1}{\log ct}\Big((1-\gamma)(k+1)m_k + 3m_{k+1}\Big) \\
+\; &\frac{1}{\log^2 ct}\Big((1 - \tfrac{1}{12}\pi^2 - \gamma + \tfrac{1}{2}\gamma^2)(k+1)(k+2)m_k \\
&\qquad - 3(1-\gamma)(k+2)m_{k+1} + 9m_{k+2}\Big)\Big]
\end{aligned}
\tag{5.5}
$$

in which $\gamma = 0.577215...$ denotes Euler's constant. Both $\overline{M(t)M'(t)}$ and the product $\overline{M(t)}\;\overline{M'(t)}$ the appear as t^2 times a power series in $1/\log ct$ of which the leading term is of order $k+k'+2$, and in which the three leading orders have to be retained. Upon carrying out the subtraction one finds that the two leading orders cancel and the correlation $\overline{\Delta M(t)\Delta M'(t)}$ appears to be proportional to $t^2/\log^{k+k'+4} ct$. Explicitly, as $t \to \infty$,

$$
\overline{\Delta M(t)\Delta M'(t)} \simeq \mathcal{A}^2(k+1)(k'+1)m_k m'_{k'}\frac{3^{k+k'+2}\,t^2}{\log^{k+k'+4} ct} \qquad (t \to \infty)
\tag{5.6}
$$

in which the coefficient $\mathcal{A}$ is given by

$$
\mathcal{A}^2 = 1 \;+\; (F_{1,3} - \tfrac{1}{6})\pi^2
\tag{5.7}
$$

Numerical evaluation gives $F_{1,3} = 0.27415...$, whence $\mathcal{A}^2 = 2.0608....$ Eq. (5.6) is the same as for the two-dimensional simple random walk [14, 7, 6, 17], but with a different constant $\mathcal{A}$.

6 Universality of fluctuations

We consider in this section the normalized deviations from average

$$
\theta_M(t) = \frac{\Delta M(t)}{\overline{\Delta M^2(t)}^{1/2}}
\tag{6.1}
$$

These random functions of time satisfy by construction

$$\overline{\theta_M(t)} = 0, \qquad \overline{\theta_M^2(t)} = 1 \tag{6.2}$$

We consider now two arbitrary observables M and M'. When $\frac{2}{3} \le \alpha < 1$ we have from Eq. (6.1) together with either Eq. (5.2) or Eq. (3.13) that

$$\overline{\theta_M(t)\theta_{M'}(t)} = 1 \tag{6.3}$$

It then follows from Eqs. (6.2) and (6.3) that the difference $\theta_M - \theta_{M'}$ is a random variable of zero average and zero variance. Such a random variable can only be itself equal to zero. We therefore deduce that, when $\frac{2}{3} \le \alpha < 1$, in the limit $t \to \infty$ all $\theta_M(t)$ are equal to a single random variable, which we will call $\Theta_\alpha(t)$, thus indicating explicitly its α dependence.

When $1 \le \alpha < 2$ we have for two observables M and M' *whose orders, k and k', are equal* from Eq. (6.1) and either Eq. (5.6) or Eq. (4.10) again the result (6.3). Hence, when $1 \le \alpha < 2$, in the limit $t \to \infty$ all $\theta_M(t)$ with $k = 0$ are equal to a single random variable – that we will call $\Theta_{\alpha 0}(t)$ –, and all $\theta_M(t)$ with $k = 1$ are similarly equal to a single random variable – that we will call $\Theta_{\alpha 1}(t)$. The variables $\Theta_\alpha(t)$, $\Theta_{\alpha 0}(t)$, and $\Theta_{\alpha 1}(t)$ are *universal* in the sense that they are independent of the observables M (but depend at most on their order).

In each of these cases the key ingredient necessary for arriving at Eq. (6.3) is the factorization of $\overline{M(t)M'(t)}$ into an M and an M' dependent part. This also explains why for $\alpha < \frac{2}{3}$ the same reasoning fails.

The *cross correlation* between $\Theta_{\alpha 0}(t)$ and $\Theta_{\alpha 1}(t)$ is easily found from the correlation between a $\theta_M(t)$ and a $\theta_{M'}(t)$ with $k = 0$ and $k' = 1$, and use of the second one of Eqs. (4.10). The answer is independent of the choice of M and M', as it had to be, and reads

$$\overline{\Theta_{\alpha 0}(t)\Theta_{\alpha 1}(t)} = \mathcal{B}_{\alpha 10}^2 (\mathcal{B}_{\alpha 00}\mathcal{B}_{\alpha 11})^{-1} \tag{6.4}$$

The coefficient ratio on the RHS of this equation depends only on the exponent α and must necessarily be less than unity. In the limit $\alpha \to 1^+$ it approaches unity and for $\alpha < 1$ the distinction between surface and bulk observables is no longer reflected in the fluctuations.

Upon combining all these conclusions we get explicitly

$$\Delta M(t) = \begin{cases} m_2\, \mathcal{C}\, (t \log t)^{\frac{1}{2}}\, \Theta_{\frac{2}{3}}(t) & (\alpha = \frac{2}{3}) \\ m_2\, \mathcal{B}_\alpha\, t^{2-\frac{1}{\alpha}}\, \Theta_\alpha(t) & (\frac{2}{3} < \alpha < 1) \\ m_k\, \mathcal{A}\, (k+1)\, t\, (\frac{1}{3}\log ct)^{-k-2}\, \Theta_1(t) & (\alpha = 1; k = 0, 1) \\ m_0\, \mathcal{B}_{\alpha 00}\, t^{\frac{1}{\alpha}}\, \Theta_{\alpha 0}(t) & (1 < \alpha < 2; k = 0) \\ m_1\, \mathcal{B}_{\alpha 11}\, t^{\frac{2}{\alpha}-1}\, \Theta_{\alpha 1}(t) & (1 < \alpha < 2; k = 1) \end{cases} \tag{6.5}$$

Here all M dependence is contained in the coefficients m_n.

7 Conclusions

We have studied a large class of properties $M(t)$ of the support of the one-dimensional t step Riemann walk. These include the number $S(t)$ of distinct sites visited, and the number $I(t)$ of sequences of visited sites. The $M(t)$ fall into two classes, the *bulk* or *S-like* properties, and the *surface* or *I-like* properties. The asymptotic laws found in the preceding sections for the averages, variances, and correlation of $S(t)$ and $I(t)$ have been summarized in Table I in the Introduction.

It appears from that table that in the exponent regime $0 < \alpha \leq 1$ the ratios $\overline{\Delta S^2(t)}^{1/2}/\overline{S(t)}$ and $\overline{\Delta I^2(t)}^{1/2}/\overline{I(t)}$ tend to zero when $t \to \infty$, which indicates that the distributions of $S(t)$ and of $I(t)$ become infinitely narrowly peaked around their average. Hence in this exponent regime the ratio $s(t) \equiv \overline{S(t)}/\overline{I(t)}$ represents the *average number of sites per visited sequence*. When α is strictly less than unity we have explicitly

$$\lim_{t\to\infty} s(t) = \frac{m_1[S]}{m_1[I]} = \frac{\hat{G}(0,1) - \hat{G}(1,1)}{\hat{G}(0,1) + \hat{G}(1,1)} \qquad (0 < \alpha < 1) \tag{7.1}$$

The first equality is based on Eq. (3.5) and in the second one we used the definition (3.4) of the $m_n[M]$ and the remarks at the end of Sec. 2.2. The finiteness of the result (7.1) means that in the large t limit every new step of the walk creates a new visited sequence with a finite nonzero probability. This explains that in this exponent regime the asymptotic power laws do not distinguish between bulk and surface properties. For $\alpha \to 1^-$ expression (7.1) diverges, and when $\alpha = 1$ the ratio $s(t)$ increases logarithmically with t.

In the exponent regime $1 < \alpha < 2$ the appropriately scaled distributions of $S(t)$ and $I(t)$ are of finite width even in the limit $t \to \infty$. The support has an "interior", bulk and surface properties have different asymptotic power laws, and $s(t) \to \infty$ as $t \to \infty$. In the terminology of critical phenomena, this regime is fluctuation dominated. In this regime the universality of fluctuations holds in a slightly weaker but at least as interesting a sense as for $0 < \alpha \leq 1$. To describe the fluctuations, not a single but *two* universal stochastic variables are needed, one applying to the bulk and the other to the surface properties. These two variables become fully correlated in the limit $\alpha \to 1^+$.

Finally we remark that when $2/\alpha$ is equal to one of the integers $2, 3, 4, \ldots,$ the asymptotic laws of Table I coincide with the ones known to hold for the simple random walk on a lattice of dimension $d = 2/\alpha$. Similarly, the universality properties for those α values have their analogs in the d-dimensional simple random walk. Hence the "rule of the effective dimensionality", which states the correspondence $\alpha \Leftrightarrow 2/d$, applies to all properties that we have studied. Of course it must break down when the comparison between the Riemann walk and the simple random walk is refined sufficiently. Also, we have not considered the borderline value $\alpha = 2$, which is special [5], and for which this rule fails.

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A Relations for the inverse sums G_A and g_A

We collect here some elementary matrix algebra relations useful for dealing with the inverse sums G_A and g_A occurring in the main text. The z dependence of these quantities plays no role. The presentation and notation are independent of the body of the paper.

Let L be an invertible $\ell \times \ell$ matrix. We define the "inverse sum" $\mathcal{I}(L)$ by

$$\mathcal{I}^{-1}(L) = \sum_{i,j} L^{-1}_{ij} \tag{A.1}$$

In the remainder α, β, and γ will denote constants.

Property 1. *Let J be the $\ell \times \ell$ matrix with all $J_{ij} = 1$, and let M be an invertible $\ell \times \ell$ matrix. Let $L = \alpha J + \gamma M$. Then*

$$\mathcal{I}(L) = \alpha + \gamma \mathcal{I}(M) \tag{A.2}$$

The proof of this relation is given in Ref. [17].

Property 2. *Let M and N be invertible matrices of dimensions $m \times m$ and $n \times n$, respectively, and let L be the block diagonal $\ell \times \ell$ matrix with blocks M and N. Then*

$$\frac{1}{\mathcal{I}(L)} = \frac{1}{\mathcal{I}(M)} + \frac{1}{\mathcal{I}(N)} \tag{A.3}$$

This follows directly from the definition (A.1). The calculation of $\mathcal{I}(L)$ for an $\ell \times \ell$ matrix may be reduced to an inversion problem of dimension less than ℓ also in certain cases where L is not block diagonal, as shown below.

Property 3. *Let J^{mn} be the $m \times n$ matrix with all elements equal to 1. Let L be $\ell \times \ell$ and of the form*

$$L = \begin{pmatrix} \gamma M & \beta J^{mn} \\ \beta J^{nm} & \gamma N \end{pmatrix} \tag{A.4}$$

Then

$$\mathcal{I}(L) = \frac{\gamma^2 \mathcal{I}(M)\mathcal{I}(N) - \beta^2}{\gamma \mathcal{I}(M) + \gamma \mathcal{I}(N) - 2\beta} \tag{A.5}$$

To prove this we rewrite L as $L = \beta J + \tilde{L}$, where J is as before and where

$$\tilde{L} = \begin{pmatrix} \gamma M - \beta J^{mm} & 0 \\ 0 & \gamma N - \beta J^{nn} \end{pmatrix} \tag{A.6}$$

From PROPERTY 1 we have that $\mathcal{I}(L) = \beta + \mathcal{I}(\tilde{L})$, after which by applying PROPERTY 2 and once more PROPERTY 1, we obtain after some rearrangement Eq. (A.5). For $\beta = 0$ Eq. (A.5) reduces to PROPERTY 2.

In this work the need for PROPERTIES 1 and 3 arises when the limit $\gamma \to 0$ has to be taken. For $\gamma = 0$ the matrices L that occur on the LHS of Eqs. (A.2) and (A.5) are no longer invertible, but these properties allow nevertheless $\mathcal{I}(L)$ to be calculated in that limit.

B Corrections to scaling

In Eq. (3.6) we have neglected the $\mathcal{O}(\mathbf{V}^2)$ terms that appear in Eq. (2.15). Since in the last step that led to Eq. (3.13) the leading order in $1 - z$ went down due to cancellations, we must now check that the $\mathcal{O}(\mathbf{V}^2)$ terms remain subdominant. In this Appendix we will write Eq. (2.13) in the simplified notation $\mathbf{G}_{A\cup(r+B)} = \mathbf{G} + \mathbf{W}$ where $\mathbf{G}$ and $\mathbf{W}$ are the first and second matrix, respectively, on the RHS of Eq. (2.13). Upon writing the inverse $\mathbf{G}^{-1}_{A\cup(r+B)}$ as a perturbation series in $\mathbf{W}$ and applying Eq. (2.4) one finds

$$\mathsf{G}^{-1}_{A\cup(r+B)} = \sum_{\ell=0}^{\infty}(-1)^{\ell} \sum_{c,c'} [\mathbf{G}^{-1}(\mathbf{W}\mathbf{G}^{-1})^{\ell}]_{cc'} \tag{B.1}$$

The $\ell = 0$ term of this series is the term shown explicitly on the RHS of Eq. (2.15), and has been the object of study in Sec. 3.2. We will show here that the terms with $\ell \geq 1$ produce, in the scaling limit, only higher order corrections to the final result. To this end we first consider $C_{MM'}(z)$ defined by Eq. (2.6). Let $R_{\ell}(z)$ denote the contribution to $C_{MM'}(z)$ from the ℓth term in Eq. (B.1). In order to estimate the order in $1 - z$ of $R_{\ell}(z)$ as $z \to 1$ we first deduce from Eqs. (2.14) and (2.25) that in the scaling limit the matrix elements of $\mathbf{W}$ behave as $V_{a,r+b} \simeq (1-z)^{\frac{2}{\alpha}-1}(b-a)\,F'(\xi)$, and that summing on r amounts to applying $(1-z)^{-\frac{1}{\alpha}} \int \mathrm{d}\xi$. This yields the asymptotic proportionality

$$R_{\ell}(z) \sim (1-z)^{-\frac{1}{\alpha}} \int \mathrm{d}\xi \; \frac{1}{G_0(z)} \Big[\frac{(1-z)^{\frac{2}{\alpha}-1} F'(\xi)}{G_0(z)} \Big]^{\ell} \tag{B.2}$$

where $G_0(z)$ represents the order in $1 - z$ of the matrix $\mathbf{G}$. When ℓ is odd, this integral vanishes by symmetry, which shows that the leading correction is of order $\mathbf{V}^2$, as anticipated. For $\xi \to 0$ we have, in virtue of Eq. (2.27), that $F'(\xi) \sim \xi^{-2+\alpha}$. Hence the ξ integral in Eq. (B.2) diverges in the origin for all $\ell \geq 1$ when $0 < \alpha < 1$. This signals that the main contribution comes from r values near the origin. The order in $1 - z$ of $R_{\ell}(z)$ may then

be estimated by introducing in the integral the cutoff $|\xi| \sim \mathrm{cst} \times (1-z)^{\frac{1}{\alpha}}$, which leads to $R_\ell(z) \sim (1-z)^0$. When $\ell \geq 1$ these additive corrections to $C_{MM'}(z)$ in Eq. (3.11) are negligible, therefore, with respect to the $(1-z)^{\frac{2}{\alpha}-3}$ term which, in the relevant exponent regime $\frac{2}{3} < \alpha < 1$, determines the final result.

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